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Three methods are examined for estimating values for grid points without data, based on data values available at grid points distributed in a staggered manner on a square grid. One method uses the average of values at surrounding points. The other two involve interpolation formulas of greater sophistication: bicubic spline interpolation and Fourier Transform interpolation. All three methods are applied to square grids on which values of transcendental functions of known wave numbers are placed in a

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staggered fashion. The completed fields from each method are spectrally analyzed to determine the effect of the completion method on the known spectrum of the incomplete field. It is determined that all three methods tend to alias high wave number components to lower wave number components. In addition, a spectral spreading is observed in the bicubic spline technique. The Fourier Transform method results in the least amount of aliasing, especially when the incomplete field is made up primarily of low wave number components.

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I. INTRODUCTION

In meteorological analysis it is often necessary to analyze fields of data values assigned to a regularly spaced network of grid points on which only a part of the data is available. This problem arises in numerous contexts, such as analysis of synoptic fields with some observations missing or in analysis of forecast fields from numerical weather prediction models at arbitrarily chosen time steps. The latter case results from the use of time iteration techniques in which values are predicted for only the even staggered (i.e., $i + j = 2k$, $k = 0, 1, 2, \ldots$, where i, j are the grid indices) grid points at one time step and only at the odd staggered (i.e., $i + j = 2k + 1$, $k = 0, 1, 2, \ldots$) grid points at the next. Various types of analyses may be desired at arbitrarily chosen time steps in order to closely observe the evolution of certain features in the field. In order to perform such analyses, values of the parameter under study must be available at each grid point at the time step chosen to be analyzed. The problem then is to estimate values for the grid points where data are missing in the way most representative of the incomplete field so that information will be neither added to nor taken from the field.

The solution of this problem requires the use of an objective interpolation technique to obtain the missing values. However, care must be taken in the choice of such a technique so that the incomplete field is not misrepresented by spurious information which may come about through the act of completing the field. An example of such distortion is the aliasing of waves of short wavelengths into longer wave lengths, therefore artificially changing the spectrum of the field at that time step. Since a spectral analysis is commonly performed on such fields in order to determine the amplitudes of the various wave components within the field, such distortion in completing the field is obviously undesirable.

The present study considers three objective methods of completing a staggered data field. A two-dimensional field with equally spaced grid points in each direction is assumed. The methods discussed are: (1) a simple algebraic average of the values at the four grid points surrounding the grid point with the missing value, or the Four Point Averaging Method (FPAM), (2) rotation of the grid axes by 45° to obtain a regular grid network of the given values and a bicubic spline interpolation using these values, or the Rotation-Bicubic Spline Method (RBSM), and (3) rotation of the axes and use of Fourier coefficients from the rotated field to generate a truncated Fourier series as the interpolation function, or the Rotation-Fourier Series Method (RFSM). In all three methods it is assumed that the field is periodic, so that the grid point values are repeated in the regions outside the grid in the same sequence in which they appear in the grid. This assumption aids in obtaining values for the boundary grid points in FPAM and in completing the rotated field in RBSM and RFSM.

II. DESCRIPTION OF COMPLETION METHODS AND PERFORMANCE TESTS

In the methods described below, a two-dimensional array of M grid points equally spaced in both directions is assumed. The two cases of data staggering considered in each completion method are (1) even staggered data, with data available only at grid points whose indices (i,j) satisfy the relation $i + j = 2k$; $k = 0, 1, 2, \ldots, M - 1$, and (2) odd staggered data, with values given only at grid points (i,j) such that $i + j = 2k + 1$; $k = 0, 1, 2, \ldots, M - 1$. The objective of each method is to use the known values to obtain representative values at the grid points where data are not available.

A. Four Point Averaging Method (FPAM)

This method calculates a value at each grid point in the field where data are missing by averaging the values at the four grid points located immediately above, below, to the right, and to the left of it in the grid. Thus $u_{i,j} = (u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1})/4$ for each grid point (i,j) with a missing value. Values are obtained for the outermost grid points with missing values by assuming the field is periodic, so that values are repeated in the regions adjacent to the grid. Thus, for the left-most grid points $u_{0,j} = (u_{M-1,j} + u_{1,j} + u_{0,j-1} + u_{0,j+1})/4$. The analogous expressions hold for computation of the missing values of the grid points on the top, bottom, and right side of the grid.

B. Rotation-Bicubic Spline Method (RBSM)

In addition to FPAM, two interpolation techniques were examined. Both methods begin with a rotation of the (i,j) grid axes by 45° counter-clockwise to obtain (I,J) grid axes of regularly spaced points. This is illustrated for both the even and odd staggered cases for M = 8 in Figure 1. The values for the grid points on the I,J axes are obtained from the given values at locations where they do not directly coincide by means of the previously stated assumption that the original field repeats itself in adjacent regions. Thus $u_{I,J} = u_{i,j}$ where for the even staggered case

$i = I - J + 1$; $i = i + M$ for $i \leq -1$, $i = i - M$ for $i > M - 1$

$j = I + J - 1$; $j = j + M$ for $j \leq -1$, $j = j - M$ for $j > M - 1$

and for the odd staggered case

$i = I - J$; $i = i + M$ for $i \leq -1$, $i = i - M$ for $i > M - 1$

$j = I + J - 1$; $j = j + M$ for $j \leq -1$, $j = j - M$ for $j > M - 1$.

Note that in the resulting configuration, the grid points with missing values are located in the centers of the grid squares of the rotated grid network.

At this point, a bicubic spline interpolation method was used in RBSM to interpolate values for the centers of the squares. The actual method used is described by DeBoor (1962)* and involves the development of a bicubic polynomial for each grid square. Thus, the interpolated value F at any point x,y on the grid, where this point lies within the grid square I,J such that $x_{I-1} \leq x \leq x_I$ and $y_{J-1} \leq y \leq y_J$, is obtained from the polynomial expression

*DeBoor, Carl, 1962: Bicubic spline interpolation. J. Math. and Phys., 41, 212-218.

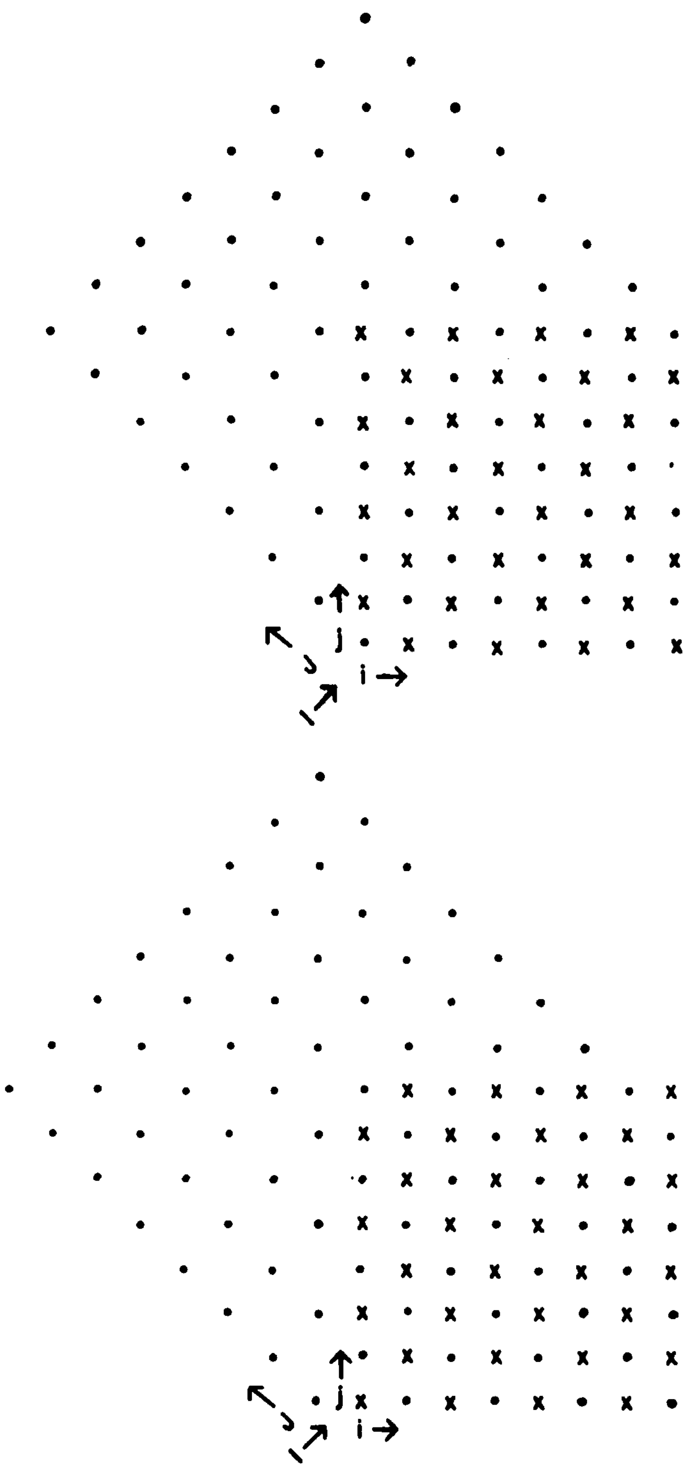

Figure 1. Rotation of the i,j axes to obtain the I,J axes for even (top) and odd (bottom) staggered grids. Grid points marked by x denote locations where original data are available.

$$F_{I,J}(x,y) = \sum_{l=0}^{3} \sum_{k=0}^{3} \psi^{I,J}_{k,l} (x - x_{I-1})^k (y - y_{J-1})^l. \quad (1)$$

The 16 values of ψ for each grid square (I,J) are made up of linear combinations of u, $\partial u/\partial x$, $\partial u/\partial y$, and $\partial^2 u/\partial x \partial y$ at the four corners of each grid square. Thus, in order to calculate the polynomial coefficients, the value of u and its three indicated derivatives must be known at each grid point on the rotated grid. DeBoor's method describes the means of obtaining the remaining values of the derivatives once the following values are known:

$$u_{I,J} \quad , \ I = 0, 1, \ldots, M-1; \ J = 0, 1, \ldots, M-1$$

$$p_{I,J} = \frac{\partial u_{I,J}}{\partial x} \ , \ I = 0, M-1; \ J = 0, 1, \ldots, M-1$$

$$q_{I,J} = \frac{\partial u_{I,J}}{\partial y} \ , \ I = 0, 1, \ldots, M-1; \ J = 0, M-1$$

$$r_{I,J} = \frac{\partial^2 u_{I,J}}{\partial x \partial y} \ , \ I = 0, M-1; \ J = 0, M-1.$$

Note that these are the values of the normal derivatives at their respective outermost grid points on the rotated grid network. These values were obtained by subjecting the rotated grid values to a Fast Fourier Transform algorithm to determine the coefficients of the truncated Fourier series. If we introduce the notation $s_m(x) = \sin(2\pi m x/L_x)$, $c_n(y) = \cos(2\pi n y/L_y)$, etc., this Fourier series is expressed in the form

$$u(x,y) = \sum_{n=0}^{M/2} \sum_{m=0}^{M/2} \{A_{m,n} c_m(x) c_n(y) + B_{m,n} c_m(x) s_n(y) + C_{m,n} s_m(x) c_n(y) + D_{m,n} s_m(x) s_n(y)\}. \quad (2)$$

Since $x = I\Delta x$, $y = J\Delta y$, and since we are considering the case $\Delta x = \Delta y$, then $L_x = L_y = M\Delta x$. If we choose $\Delta x = \Delta y = 1$, then

$$p_{I,J} = -P \sum_{n=0}^{M/2} \sum_{m=0}^{M/2} m \{A_{m,n} s_m(I) c_n(J) + B_{m,n} s(I) s_n(J) - C_{m,n} c_m(I) c_n(J) - D_{m,n} c_m(I) s_n(J)\}$$

$$q_{I,J} = -P \sum_{n=0}^{M/2} \sum_{m=0}^{M/2} n \{A_{m,n} c_m(I) s_n(J) - B_{m,n} c_m(I) c_n(J) + C_{m,n} s_m(I) s_n(J) - D_{m,n} s_m(I) c_n(J)\}$$

$$r_{I,J} = P^2 \sum_{n=0}^{M/2} \sum_{m=0}^{M/2} mn \Big\{ A_{m,n} s_m(I) s_n(J) - B_{m,n} s_m(I) c_n(J)$$

$$- C_{m,n} c_m(I) s_n(J) + D_{m,n} c_m(I) c_n(J) \Big\}$$

where $P = \frac{2\pi}{M}$ and $s_m(I) = \sin(PmI)$, $c_n(J) = \cos(PnJ)$, etc. These expressions are used to obtain the respective values of $p_{I,J}$, $q_{I,J}$, and $r_{I,J}$ only at the grid locations mentioned above. These values are then used along with the values of $u_{I,J}$ to determine the remaining values of $p_{I,J}$, $q_{I,J}$, and $r_{I,J}$ through four sets of systems of algebraic equations which are solved by Gauss elimination. A description of the system of equations and a discussion of the technique used in their solution is given in some detail by DeBoor and will not be reproduced here.

Once the values of $u_{I,J}$ and its derivatives are known at all grid points on the rotated grid, the values of the interpolation coefficient ψ for each grid square (I,J) are determined from the matrix equation

$$\psi^{I,J} = A(\Delta x_{I-1})\, K_{I,J} A^T(\Delta y_{J-1})$$

where $K_{I,J} = \begin{vmatrix} B_{I-1,J-1} & B_{I-1,J} \\ B_{I,J-1} & B_{I,J} \end{vmatrix}$ with $B_{v,w} = \begin{vmatrix} u_{v,w} & q_{v,w} \\ p_{v,w} & r_{v,w} \end{vmatrix}$

and

$$A(h) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3/h^2 & -2/h & 3/h^2 & -1/h \\ 2/h^3 & 1/h & -2/h^2 & 1/h \end{vmatrix}$$

where A^T represents the transpose of the matrix A. Since we have chosen $\Delta x = \Delta y = 1$ for all squares, the matrix A is the same for all computations. The values of $\psi^{I,J}_{k,l}$ are the elements in the kth row and lth column of the resulting product matrix. Since the required value corresponds to the center of each grid square, the interpolation expression for each square I,J reduces to

$$F_{I,J} = \sum_{l=0}^{3}\sum_{k=0}^{3} \psi_{k,l}^{I,J}(.5)^k(.5)^l.$$

Then $u_{i,j} = F_{I,J}$ where the relationships

$i = I - J + 1$; $i = i + M$ for $i \leq -1$, $i = i - M$ for $i > M - 1$

$j = I + J$; $j = j + M$ for $j \leq -1$, $j = j - M$ for $j > M - 1$

for the even staggered case, and

$i = I - J$; $i = i + M$ for $i \leq -1$, $i = i - M$ for $i > M - 1$

$j = I + J$; $j = j + M$ for $j \leq -1$, $j = j - M$ for $j > M - 1$

for the odd staggered case are used to locate the interpolated values at their proper location in the original grid. The resulting values $u_{i,j}$, $i = 0, 1, \ldots, M - 1$; $j = 0, 1, \ldots, M - 1$ represent the values of the completed field for the parameter u.

C. Rotation-Fourier Series Method (RFSM)

This method uses the rotated grid values obtained in the manner described above and uses the truncated Fourier series (2) to interpolate values at the centers of the grid squares. First, the rotated grid values are subjected to the Fast Fourier Transform algorithm as in RBSM, and the coefficients $A_{m,n}$, $B_{m,n}$, $C_{m,n}$ and $D_{m,n}$ for $m = 0, 1, \ldots, M/2$; $n = 0, 1, \ldots M/2$ are obtained. Then the interpolated value F for the center of each grid square (I,J) is given by the expression

$$F_{I,J} = \sum_{n=0}^{M/2}\sum_{m=0}^{M/2}\Big\{A_{m,n}c_m(I+\tfrac{1}{2})c_n(J+\tfrac{1}{2}) + B_{m,n}c_m(I+\tfrac{1}{2})s_n(J+\tfrac{1}{2}) + C_{m,n}s_m(I+\tfrac{1}{2})c_n(J+\tfrac{1}{2}) + D_{m,n}s_m(I+\tfrac{1}{2})s_n(J+\tfrac{1}{2})\Big\} \quad (3)$$

where again we have chosen $\Delta x = \Delta y = 1$. Thus $u_{i,j} = F_{I,J}$ for the indices (i,j) of the grid points with missing values, and the interpolated values are located at their proper location in the original grid in a manner similar to that used in RBSM.

After each method was formulated, it was tested to determine its performance characteristics. Of particular interest was the manner in which each method might distort periodic variations existent in the original complete field. To determine this effect quantitatively for each method, the values of products of transcendental functions at the discrete intervals were used as the original grid point values. For example, values for the product cos $(2\pi mi/M)$ cos $(2\pi nj/M)$ $i = 0, 1, 2, \ldots, M-1$; $j = 0, 1, 2, \ldots, M-1$ for various values of $m,n \leq M/2$ were used for the grid point values in one set of runs for each method. However, only the values of these functions at either even or odd grid points were used as inputs for the computations, and the respective methods were used to fill in the missing values. Computations were carried out for only one value of M ($M = 16$), but it is believed that any even value for M will yield the corresponding results.

III. RESULTS AND DISCUSSION

The results of the model tests are interpreted from the harmonic analyses of the fields completed by each method. Since the values of the coefficients of the Fourier terms which make up the complete input field are known, the ability of each method to replicate this field from the staggered data is determined by comparing the known coefficients with the values of the Fourier coefficients which represent the completed field. Any departure of the coefficients that describe the completed field from the coefficients of the original complete field represents a harmonic distortion brought about by the completion method used.

The effect of FPAM was to partially distort the component (m,n) present in the staggered input field for all values of $m,n > 0$. The distortion always occurred in such a way that a portion of the amplitude of the component that was present in the original field was lost to the component $\left[(M/2) - m,\ (M/2) - n\right]$, which shall be referred to as the complementary component of (m,n). This distortion can be viewed more quantitatively by considering the use of FPAM in completing the field made up of the even or odd staggered values of the function $\phi(i,j)$ defined by

$$\phi(i,j) = \sum_{n=0}^{M/2} \sum_{m=0}^{M/2} \Big\{ \alpha_{m,n} c_m(i) c_n(j) + \beta_{m,n} c_m(i) s_n(j) + \gamma_{m,n} s_m(i) c_n(j) + \delta_{m,n} s_m(i) s_n(j) \Big\} \quad (4)$$

for i, j = 0, 1, 2, ..., M - 1 and the arbitrary (M/2) + 1 by (M/2) + 1 matrices α, β, γ and δ. Since FPAM calculates the missing values in the grid by means of a linear combination of the known values, it can be thought of as a linear operator in completing the field. When FPAM is used to complete a staggered data field composed of a linear combination of several components, the results are equivalent to the sum of the fields of the individual components as completed by FPAM. This was verified in several test runs. For this reason, the results of the completion of the staggered values from (4) above could be inferred from the results of completing fields represented by individual terms in the series. The Fourier coefficients representing the field that would result from the completion of the even staggered values from (4) are:

$$A_{m,n} = \epsilon_{m,n}(\alpha_{m,n} + \alpha_{(M/2)-m,\ (M/2)-n})$$

$$B_{m,n} = \epsilon_{m,n}(\beta_{m,n} - \beta_{(M/2)-m,\ (M/2)-n})$$

$$C_{m,n} = \epsilon_{m,n}(\gamma_{m,n} - \gamma_{(M/2)-m,\ (M/2)-n})$$

$$D_{m,n} = \epsilon_{m,n}(\delta_{m,n} + \delta_{(M/2)-m,\ (M/2)-n}).$$

The coefficients which represent the field completed from the odd staggered values are:

$$A_{m,n} = \epsilon_{m,n}(\alpha_{m,n} - \alpha_{(M/2)-m,\ (M/2)-n})$$

$$B_{m,n} = \epsilon_{m,n}(\beta_{m,n} + \beta_{(M/2)-m,\ (M/2)-n})$$

$$C_{m,n} = \epsilon_{m,n}(\gamma_{m,n} + \gamma_{(M/2)-m,\ (M/2)-n})$$

$$D_{m,n} = \epsilon_{m,n}(\delta_{m,n} - \delta_{(M/2)-m,\ (M/2)-n}).$$

In these expressions, $A_{m,n}$, $B_{m,n}$, $C_{m,n}$, and $D_{m,n}$ represent the coefficients of the truncated Fourier series (2) for the completed field and $\epsilon_{m,n}$ is a measure of the distortion of the (m,n) component present in the staggered field and shall be referred to as the response function for FPAM. Values of $\epsilon_{m,n}$ for selected values of (m,n) are displayed in Table 1.

TABLE 1. Values of Response Function $\epsilon_{m,n}$ for FPAM, M = 16.

n									
8		.4810		.3457		.1543		.0190	
7	.5190	.5000	.4458	.3647	.2690	.1734	.0923	.0381	.0190
6		.5542		.4189		.2276		0923	
5	.6543	.6353	.5811	.5000	.4043	.3087	.2276	.1734	.1543
4		.7310		.5957		.4043		.2690	
3	.8457	.8266	.7725	.6913	.5957	.5000	.4189	.3647	.3457
2		.9078		.7725		.5811		.4458	
1	.9810	.9620	.9078	.8266	.7310	.6353	.5542	.5000	.4810
0		.9810		.8457		.6543		.5190	
	0	1	2	3	4	5	6	7	8

m

Note that $\epsilon_{m,n} = \epsilon_{n,m}$ for all (m,n), indicating the symmetry resulting from the use of a square grid. The value of $\epsilon_{m,n}$ represents that fraction of the amplitude of the harmonic (m,n) present in the original field which is preserved in the completed field and the fraction of the amplitude of its complementary harmonic which is aliased in such a way as to result in constructive (+) or destructive (-) interference with the wave with components (m,n). Note also that $\epsilon_{m,n} + \epsilon_{(M/2)-m,\ (M/2)-n} = 1$ for all (m,n); thus the complementary harmonic has the complementary fraction of its amplitude preserved in FPAM. Table 1 shows that this fraction decreases with increasing value of the sum m + n, or with increasing value of one of the two components when the other is held constant. Thus, a larger fraction of the amplitudes of the longer waves (smaller wave components) than those of shorter waves are preserved when FPAM is used to complete the field. Only a small amount of their amplitude is lost to the complementary short wave. In contrast, the shorter waves are heavily aliased to their long wave complements.

The relationship between the results for the completion of the even staggered field and those for the odd staggered field comes about from the way complementary harmonics combine to make up the even and odd staggered values. To illustrate this, consider the values of the sum of the two complementary components

$$\alpha_{m,n}\cos(Pmi)\cos(Pnj) + \alpha_{(M/2)-m,\ (M/2)-n}\cos\{P[(M/2 - m]i\}$$
$$\times \cos\{P[(M/2) - n]j\}.$$

It can be shown that this sum is equivalent to

$$[\alpha_{m,n} + \alpha_{(M/2)-m,\ (M/2)-n}\cos(\pi i)\cos(\pi j)] \cos Pmi \cos Pnj.$$

The factor $\cos(\pi i)\cos(\pi j)$ is positive for even staggered values of (i,j), and negative for odd staggered values. The term in brackets represents the complete amplitude of the (m,n) component of the cos•cos term within the staggered field, and it is actually this amplitude that is modified by FPAM. Since all of the data values used to generate the missing values are represented by this expression, the sign in the amplitude is preserved in the completed field. Thus, the entire completed field is made up of components of this form with an amplitude modification, as can be seen in the Fourier components of the field as completed by FPAM. The same sign orientation can be shown to hold for the sin•sin terms, while the opposite relationship (amplitudes additive for odd staggered grid points, subtractive for even staggered grid points) exists for the cos•sin and sin•cos terms in the Fourier expansion. Since complementary components combine in this way to represent the even and odd staggered values on which the completed fields are based, regardless of which completion method is used, the Fourier coefficients representing the completed fields using any method will have this general form.

This fact is borne out in the results of the use of RBSM to complete the staggered data field. Again, if we use the even and odd staggered values of (4) as our incomplete fields, we find that the field completed from the even staggered data has the Fourier components

$$A_{m,n} = \nu_{m,n}(\alpha_{m,n} + \alpha_{(M/2)-m,\ (M/2)-n}) + \Gamma^{e}_{A_{m,n}}$$

$$B_{m,n} = \nu_{m,n}(\beta_{m,n} - \beta_{(M/2)-m,\ (M/2)-n}) + \Gamma^{e}_{B_{m,n}}$$

$$C_{m,n} = \nu_{m,n}(\gamma_{m,n} - \gamma_{(M/2)-m,\ (M/2)-n}) + \Gamma^{e}_{C_{m,n}}$$

$$D_{m,n} = \nu_{m,n}(\delta_{m,n} + \delta_{(M/2)-m,\ (M/2)-n}) + \Gamma^{e}_{D_{m,n}}$$

and the completed odd staggered field has the components

$$A_{m,n} = \nu_{m,n}(\alpha_{m,n} - \alpha_{(M/2)-m,\ (M/2)-n}) + \Gamma^{o}_{A_{m,n}}$$

$$B_{m,n} = \nu_{m,n}(\beta_{m,n} + \beta_{(M/2)-m,\ (M/2)-n}) + \Gamma^{o}_{B_{m,n}}$$

$$C_{m,n} = \nu_{m,n}(\gamma_{m,n} + \gamma_{(M/2)-m,\ (M/2)-n}) + \Gamma^{o}_{C_{m,n}}$$

$$D_{m,n} = \nu_{m,n}(\delta_{m,n} - \delta_{(M/2)-m,\ (M/2)-n}) + \Gamma^{o}_{D_{m,n}}$$

In this case, the response function can be separated into a sum of two separate factors. The factor $\nu_{m,n}$, displayed as a function of m and n in Table 2, is the primary response

TABLE 2. Primary Response Function $\nu_{m,n}$ for RBSM, M = 16

n \ m	0	1	2	3	4	5	6	7	8
8		.3548		.0767		.0067		.0001	
7	.6452	.5000	.2528	.1147	.0430	.0135	.0034	.0006	.0001
6		.7472		.2334		.0400		.0034	
5	.9233	.8853	.7666	.5000	.2316	.1080	.0400	.0135	.0067
4		.9570		.7684		.2316		.0430	
3	.9934	.9865	.9601	.8920	.7684	.5000	.2334	.1147	.0767
2		.9966		.9601		.7666		.2548	
1	.9999	.9995	.9966	.9865	.9570	.8853	.7472	.5000	.3548
0		.9999		.9934		.9233		.6452	

function and behaves in a manner similar to the response function $\epsilon_{m,n}$ from FPAM. However, when even or odd staggered values from a single Fourier term are used as input, non-zero Fourier coefficients for other than just that component (m,n) and its complement are involved in the resulting field. For example, appreciable values for the Fourier coefficients $B_{m,n}$, $C_{m,n}$, and $D_{m,n}$ in addition to $A_{m,n}$ existed for many values of (m,n) within the completed field when even or odd values for cos $(2\pi mi/M)$cos $(2\pi nj/M)$ were used in the incomplete field. This indicates a sort of spectral spreading of the harmonic originally existent in the input field by RBSM, which was not at all evident in FPAM. The factor Γ in the expressions represents this distortion. The nature of this departure in both magnitude and phase distortion was investigated by using values of the function cos $(2\pi ki/M)$ cos $(2\pi lj/M)$ for various integer values of k, $l < M/2$ at even staggered grid points. These incomplete fields were then completed using RBSM. For each pair of values (k,l) used, the completed field is given by the expression

$$\begin{aligned}
\phi(i,j) = &\left[\nu_{k,l} + (1 - \nu_{k,l})\cos(\pi i)\cos(\pi j)\right] c_k(i)c_l(j) \\
&+ \left[1 - \cos(\pi i)\cos(\pi j)\right] \times \left\{ \sum_{n=0}^{l-1} \sum_{m=0}^{k-1} \left[\xi_{A_{m,n}} c_m(i)c_n(j) \right.\right. \\
&+ \xi_{B_{m,n}} c_m(i)s_n(j) + \xi_{C_{m,n}} s_m(i)c_n(j) \\
&\left. + \xi_{D_{m,n}} s_m(i)s_n(j) \right] + \xi_{B_{k,l}} c_k(i)s_l(j) \\
&+ \xi_{C_{k,l}} s_k(i)c_l(j) + \xi_{D_{k,l}} s_k(i)s_l(j) \\
&+ \sum_{n=l+1}^{(M/4)-1} \sum_{m=k+1}^{(M/4)-1} \left[\xi_{A_{m,n}} c_m(i)c_n(j) + \xi_{B_{m,n}} c_m(i)s_n(j) \right. \\
&\left.\left. + \xi_{C_{m,n}} s_m(i)c_n(j) + \xi_{D_{m,n}} s_m(i)s_n(j) \right] \right\} \qquad (5)
\end{aligned}$$

for $k + l \neq M/2$, and by

$$\phi(i,j) = \tfrac{1}{2}\left[1 + \cos(\pi i)\cos(\pi j)\right] c_k(i)c_l(j) \qquad (6)$$

for $k + l = M/2$ which is identically zero at odd staggered grid points. Note that both expressions reduce to $\cos(2\pi ki/M)$ $\cos(2\pi lj/M)$ at even staggered grid points as is expected since these were the values given as input. The form of (5) comes about because it was observed from the Fourier analysis of the resulting field that for all cases where $k + l \neq M/2$ and for all $m,n \leq (M/2) - 1$,

$$\xi_{A_{m,n}} = -\xi_{A_{(M/2)-m,\ (M/2)-n}}$$

$$\xi_{B_{m,n}} = \xi_{B_{(M/2)-m,\ (M/2)-n}}$$

$$\xi_{C_{m,n}} = \xi_{C_{(M/2)-m,\ (M/2)-n}}$$

$$\xi_{D_{m,n}} = -\xi_{D_{(M/2)-m,\ (M/2)-n}}.$$

The value of twice the quantity in $\{\ \}$ in (5) was obtained at all odd staggered grid points by subtracting the value of the first term in (5) from the completed field values at all odd grid points. The magnitude of the resulting field of odd staggered departure values was evaluated by averaging their absolute values over the grid. The average absolute values from each completed grid for odd values of l are displayed in Table 3. The magnitude of this departure of the

TABLE 3. Average Absolute Value of Secondary Distortion Factor Obtained from Using RBSM on Even Staggered Values of $\cos(2\pi ki/M)\cdot\cos(2\pi lj/M)$

l								
7	.00002	.04831	.02138	.00695	.00192	.00034	.00004	.00005
5	.02209	.05192	.00002	.04906	.01515	.00768	.00206	.00078
3	.00206	.00768	.01515	.04906	.00002	.05192	.02209	.01563
1	.00004	.00034	.00192	.00695	.02139	.04831	.00002	.05850
	1	2	3	4	5	6	7	8

k

generated values from the primary response function $\nu_{k,l}$ is largest for values of (k,l) near k + l = M/2 and decreases with increasing value of $|(M/2) - (k + l)|$. Its value drops to zero at k + l = M/2 as is indicated by (6). The odd staggered departure values were plotted on a horizontal grid, and shading was added to distinguish the regions of positive and negative values and thus indicate the distribution of phase of the departure component. The results of this analysis for four pairs of values for (k,l) are shown in Figure 2. The existence of several wave numbers in both directions is evident in each diagram, again indicating a spectral spreading of the original component over a narrow band of wave components. No clear correlation of the width of this band with values of k + l was observed, in contrast with the variation in the amplitude within the departure fields with k + l as shown in Table 3. However, the monotonic variation of the primary distortion factor $\nu_{m,n}$ from RBSM in contrast with the variation in magnitude of this secondary distortion with k + l indicates the distinctly separate nature of these two distortions which come about in the use of this method.

The results from the data grid completion using the RFSM have the general form common to all methods as discussed. previously. Again, using the even staggered values from (4) on an M x M data grid, the RFSM yields a completed field with the Fourier coefficient values

$$A_{m,n} = \begin{cases} \alpha_{m,n} + \alpha_{(M/2)-m,\ (M/2)-n} & ;\ m + n < M/2 \\ \frac{1}{2}(\alpha_{m,n} + \alpha_{(M/2)-m,\ (M/2)-n}) & ;\ m + n = M/2 \\ 0 & ;\ m + n > M/2 \end{cases}$$

$$B_{m,n} = \begin{cases} \beta_{m,n} - \beta_{(M/2)-m,\ (M/2)-n} & ;\ m + n < M/2 \\ \frac{1}{2}(\beta_{m,n} - \beta_{(M/2)-m,\ (M/2)-n}) & ;\ m + n = M/2 \\ 0 & ;\ m + n > M/2 \end{cases}$$

$$C_{m,n} = \begin{cases} \gamma_{m,n} - \gamma_{(M/2)-m,\ (M/2)-n} & ;\ m + n < M/2 \\ \frac{1}{2}(\gamma_{m,n} - \gamma_{(M/2)-m,\ (M/2)-n}) & ;\ m + n = M/2 \\ 0 & ;\ m + n > M/2 \end{cases}$$

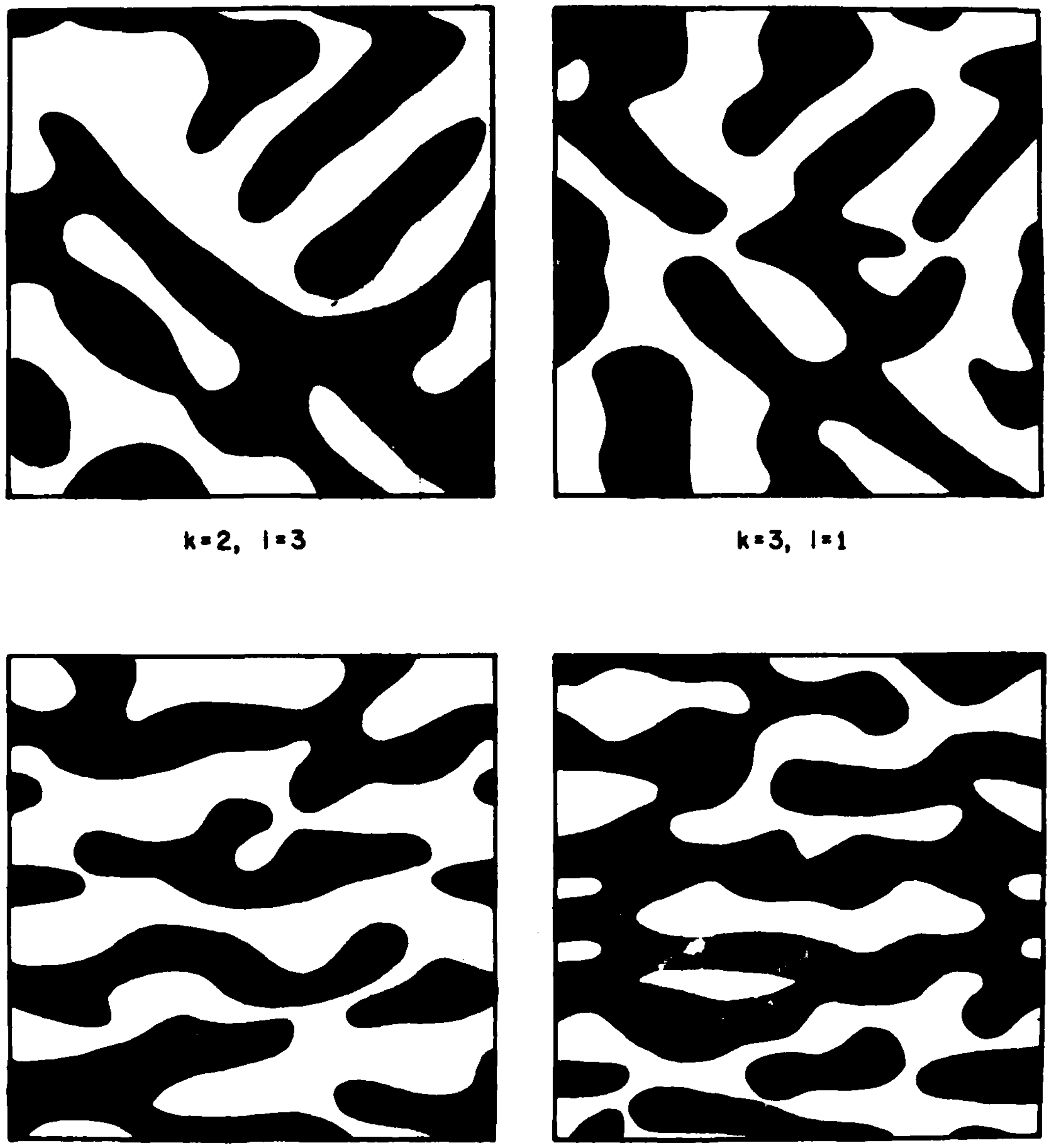

Figure 2. Distribution of phase in the secondary distortion factor from RBSM. Shaded areas are regions where distortion factor has a positive value; unshaded regions indicate negatives.

$$D_{m,n} = \begin{cases} \delta_{m,n} + \delta_{(M/2)-m,\ (M/2)-n} & ;\ m + n < M/2 \\ \frac{1}{2}(\delta_{m,n} + \delta_{(M/2)-m,\ (M/2)-n}) & ;\ m + n = M/2 \\ 0 & ;\ m + n > M/2 \end{cases}$$

while the completion of the odd staggered field yields the coefficient values

$$A_{m,n} = \begin{cases} \alpha_{m,n} - \alpha_{(M/2)-m,\ (M/2)-n} & ;\ m + n < M/2 \\ \frac{1}{2}(\alpha_{m,n} - \alpha_{(M/2)-m,\ (M/2)-n}) & ;\ m + n = M/2 \\ 0 & ;\ m + n > M/2 \end{cases}$$

$$B_{m,n} = \begin{cases} \beta_{m,n} + \beta_{(M/2)-m,\ (M/2)-n} & ;\ m + n < M/2 \\ \frac{1}{2}(\beta_{m,n} + \beta_{(M/2)-m,\ (M/2)-n}) & ;\ m + n = M/2 \\ 0 & ;\ m + n > M/2 \end{cases}$$

$$C_{m,n} = \begin{cases} \gamma_{m,n} + \gamma_{(M/2)-m,\ (M/2)-n} & ;\ m + n < M/2 \\ \frac{1}{2}(\gamma_{m,n} + \gamma_{(M/2)-m,\ (M/2)-n}) & ;\ m + n = M/2 \\ 0 & ;\ m + n > M/2 \end{cases}$$

$$D_{m,n} = \begin{cases} \delta_{m,n} - \delta_{(M/2)-m,\ (M/2)-n} & ;\ m + n < M/2 \\ \frac{1}{2}(\delta_{m,n} - \delta_{(M/2)-m,\ (M/2)-n}) & ;\ m + n = M/2 \\ 0 & ;\ m + n > M/2. \end{cases}$$

These results were inferred from using single Fourier terms to generate the incomplete fields and using the fact of the linearity of the completion operator as was done in FPAM. They are similar in nature to those obtained from FPAM. The only difference is that instead of a continuous response function as in the case of FPAM, here we have a response function with only three values over the entire range of values of the sum $m + n$. For values of $m + n < M/2$, the entire amplitude of the complementary wave component is aliased to either add to or subtract from the amplitude of the wave with components m,n. For the sum $m + n = M/2$, the response is identical to that of FPAM - exactly half of the amplitude of any component is preserved, and half of the amplitude of its complementary component is either added to or subtracted from it. Finally, for components (m,n) such that

$m + n > M/2$, the entire wave is aliased to its complement which always has components whose sum is less than $M/2$.

In all three methods, partial or total aliasing of short waves to their long wave complements occurs. We have discussed the fact that this is unavoidable because of the way complementary components combine when using the sums of their values at even or odd staggered grid points to make up the incomplete grid. This effect would also occur in the completion of an observed meteorological field in which values of the variable are available only at staggered grid points, since any natural field is made up of a sum of many pairs of complementary components when it is examined according to a particular spatial resolution on a limited-area grid. However, in RBSM an additional distortion that was not obvious in the other two methods appears in the form of the spreading out of a component present in the incomplete field over a narrow band of harmonics.

Since naturally occurring meteorological fields are made up of an infinite number of components, it is evident that the use of such methods in completing staggered meteorological data fields will result in some aliasing of short waves into longer waves. For an $M \times M$ data grid, the amplitudes of waves with components (m,n) such that $m + n < M/2$ will be increased at the expense of the shorter waves. However, in most meteorological fields, the amplitudes of the longest waves are usually the largest, and the amplitudes decrease with increasing wave number. For example, usually the largest variations in the pressure field occur on the synoptic scale, with mesoscale and microscale pressure perturbations usually being smaller in amplitude (except possibly when areas of severe storms lie within the region of interest). For this reason, there is some advantage in the use of RFSM to complete such a field in that the only distortion occurring to longer waves is the addition or subtraction of the amplitude of the complementary short waves. No fractional distortion occurs in waves with components (m,n)

such that $m + n < M/2$. If the amplitude of the short wave is small, the distortion of the amplitude of the long wave is minimized, while the short wave is aliased out altogether. While this complete removal of waves with components (m,n) such that $m + n > M/2$ must be considered a disadvantage to the use of RFSM, it is possible that the longest of these waves could be preserved in the completed field if greater spatial resolution were employed in the grid. However, they will be somewhat modified by the waves just shorter than the new value of the critical wave number criterion ($m + n = M/2$).

IV. SUMMARY AND CONCLUSIONS

The results of this study may be summarized with the aid of the schematic diagram in Figure 3. In this diagram, the wave number in the x direction of the grid (m) is on the abscissa, while the y wave number (n) is on the ordinate. The three wave number pair categories are indicated in the figure. When $m + n < M/2$, the waves present in the original complete field are resolvable in the staggered field. When $m + n = M/2$, the variations are only partially resolvable on the staggered grid. For cases where $m + n > M/2$, waves which can be resolved in the complete field cannot be resolved on the staggered field because the number of grid points in each direction is halved. These facts greatly influenced the ability of the methods to complete the fields correctly.

When RFSM was used to complete the staggered fields derived from the complete fields in which only the long waves ($m + n < M/2$) existed, no aliasing was observed. When FPAM and RBSM were used, a fraction of the amplitude of each of these harmonics was aliased to their complementary $[(M/2) - m, (M/2) - n]$ components, and this fraction increased with the sum $m + n$. The balance of the amplitude was retained by the original component (m,n). In addition, some appreciable amplitudes were observed in other components in the case of RBSM, indicating an introduction of spurious information into the completed field. For cases in which $m + n = M/2$, all three methods completed the fields in the same

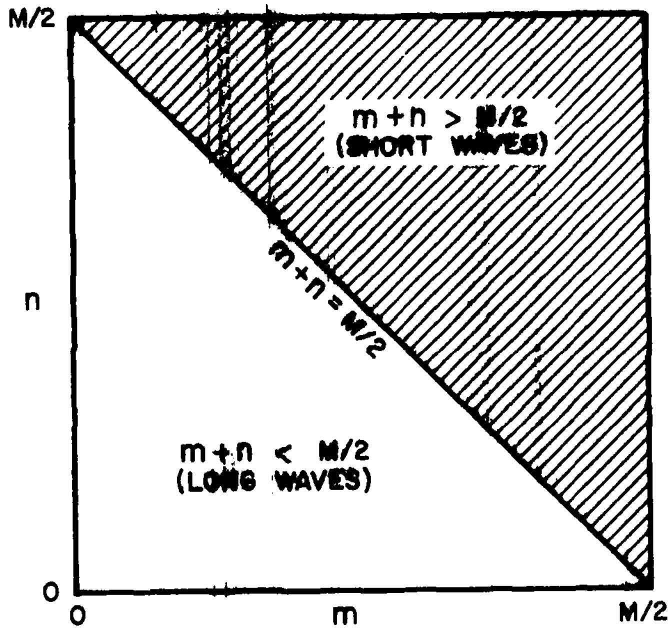

Figure 3. Schematic diagram of wave number space; m is wave number in x direction, n is wave number in y direction.

manner: half of the amplitude of the component was preserved in the completed field, and half was aliased to its complement. Finally, in cases where only the short waves ($m + n > M/2$) were present in the original field, the entire amplitude of the (m,n) component was aliased to its complement when RFSM was employed. In contrast, FPAM and RBSM showed an aliasing of only part of the amplitude to the complementary component as they did in the $m + n < M/2$ cases. The amount aliased in the $m + n > M/2$ cases continued to increase with the sum $m + n$; in fact, the amount of amplitude retained in these components is exactly the amount that had been lost by their complements through aliasing in the $m + n < M/2$ cases. As was observed in the $m + n < M/2$ cases, RBSM introduced artificial amplitude values for other harmonics.

Because the short waves ($m + n > M/2$) are unresolvable on the staggered grid, none of the methods was able to replicate the fields which contained these small scale variations. RFSM had a perfect record of completing the fields as long as only long waves ($m + n < M/2$) were present, while the other methods had varying degrees of success even with these large scale variations. These results show that the Rotation-Fourier Series Method can be successfully used to complete staggered data fields when it is certain that only these long waves are present in the sample under consideration. If the spectrum of the data field contains variations of numerous spatial scales, as is true in most cases, aliasing from short waves to long waves will occur in the process of completing the field. Without knowing the spectrum of the original field, it is impossible to ascertain the exact spectral distortion that will occur. The completed field must be considered at best an estimate of the true field from which the staggered values were derived.

V. REFERENCE

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